Re-examination of Eigenvector Derivatives

K. B. Lim* and J. L. Junkins†
Texas A&M University, College Station, Texas
and
B. P. Wang‡
University of Texas, Arlington, Texas

Analytical expressions for eigenvector derivatives for general non-self-adjoint systems using a *modal* expansion approach have not been correctly derived in several papers and books that address this problem. A common mistake in several developments on deriving eigenvector derivatives (or its perturbation forms) has been to ignore the eigenvector in question, in the expansion of its derivative, although the remaining eigenvectors forming the basis are generally non-orthogonal. This assumption is based upon explicit or implicit heuristic arguments that only directional changes contribute to eigenvector sensitivity. It is shown in this paper that the above assumption and the resulting equations are incorrect, except for the classical and most commonly encountered case of a self-adjoint problem having orthogonal eigenvectors. For the non-self-adjoint case, certain basis coefficients in the eigenvector derivative expansion have not been resolved correctly in the literature. A careful re-examination of the eigenvalue problem reveals that the two independent sets of normalizations (required for uniquely specifying the right and left eigenvectors) can be used to uniquely determine the basis coefficients. The solution derived herein for the eigenvector derivatives is shown to have generally nonzero projections onto all eigenvectors. It is also shown that basis coefficients for left and right eigenvector derivatives are related by a simple expression. A numerical example is included to demonstrate the present formulation for eigenvector derivatives with respect to a scalar parameter. We also extend Nelson's algebraic approach (for self-adjoint eigenvalue problems) to the general non-self-adjoint problem and the modal truncation approach to approximate eigenvector derivatives for the non-self-adjoint case.

Introduction

THE usefulness of modal sensitivities for analysis and design of engineering systems is well known. Some specific applications include identification of dynamical systems,[1,2] redesign of vibratory systems,[3-5] and design of control systems by pole placement.[6-8] In the above algorithms, eigenvalue and eigenvector derivatives with respect to design parameters are often required. We present here a brief literature survey on the development of derivative formulations using modal expansion or algebraic approaches. Wilkinson[9] and Meirovitch[10] present clear derivations of first-order perturbation equations for a general real matrix. Wittrick[11] derived the first derivatives of eigenvalues, while Fox and Kapoor[12] extended these results to include the first derivatives of eigenvectors for real symmetric systems. Although the first general expressions for eigenvalue and eigenvector derivatives for non-self-adjoint systems via modal expansion appears to be given by Plaut and Huseyin,[13] it is clear that the work by Rogers[14] precedes the former work. In particular, it appears that Rogers was the first to recognize the need for two sets of normalization conditions for nonsymmetric eigensystems; this truth has been largely ignored in subsequent work, leading to some confusion in the literature on eigenvector derivatives. In fairness to these authors,[13,14] their formulations are correct but incomplete. In essence, the present paper provides the completion of their developments. Additionally, we present a new relationship between left and right eigenvector derivatives.

The above-mentioned methods all utilize the modal approach for computing eigenvector derivatives. For extremely large systems, these approaches may not be very desirable. During the last decade, several "algebraic" methods[15,16] have emerged for computing eigenvector derivatives; these require only the knowledge of the eigenvector being differentiated. However, these algebraic methods require the solution of auxiliary sets of linear equations that may be ill-conditioned, thus requiring careful attention from the user. An improved modal method[17] has also been developed for the self-adjoint generalized eigenproblem in structural dynamics. Several methods of computing eigenvector derivatives have also been reported recently by Sutters et al.[18] They conclude that the improved modal method[17] is competitive with Nelson's algebraic method[15] vis-a-vis efficiency for many applications. A recent paper by Adelman and Haftka[19] discusses additional literature on eigenvector derivatives.

Numerical Example

We demonstrate here, using a (5×5) real matrix, the calculation of right and left eigenvector derivatives with respect to a single parameter. The randomly chosen matrix is

$$A(\rho) = \begin{bmatrix} 1.7 & 3.2 & 4. & 2. & -1. \\ 3.6 & .2 & (2+\rho) & -.9 & .1 \\ 1.7 & 0 & -3.5 & 9.4 & 2.9 \\ -.6 & 2. & 5. & 0 & -.3 \\ 0 & -3.9 & 1.2 & -4.5 & 1.1 \end{bmatrix}$$

where ρ is the scalar parameter appearing linearly only in location (2,3) of matrix A. The nominal matrix was chosen as the above matrix A when $\rho = 0$. It follows that the partial derivative of A with respect to ρ is zero except at location (2,3) where it equals unity.

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*Research Associate, Department of Aerospace Engineering (currently Engineering Specialist, PRC Kentron Inc., Hampton, VA). Member AIAA.

†TEES Chair Professor, Department of Aerospace Engineering. Fellow AIAA.

‡Associate Professor, Department of Mechanical Engineering. Member AIAA.

The nominal right and left eigenvalue problems of Eqs. (1) and (2) were solved and the eigenvectors normalized by Eqs. (12) and (13) to unity. By using the above normalized nominal eigenvectors, the right eigenvector derivatives were computed using Eqs. (16–18) and the left eigenvector derivatives using Eqs. (19), (20), and (22). Furthermore, the right and left eigenvector derivatives were computed for two cases, namely, "with" $\{\alpha_{ii} \neq 0, \gamma_{ii} \neq 0, i = 1,\ldots,5\}$ and "without" $\{\alpha_{ii} = 0, \gamma_{ii} = 0, i = 1,\ldots,5\}$ the diagonal terms in the eigenvector expansion equations (16) and (19).

To provide a basis for comparison, the above-computed eigenvector derivatives are compared to each other and to eigenvector derivatives computed using finite differences. The ith finite-difference eigenvector derivatives are computed as

$$\frac{\partial x_i}{\partial \rho}(\rho_0, \Delta\rho) \cong \frac{x_i(\rho) - x_i(\rho_0)}{\Delta\rho}$$

where $\Delta\rho = \rho - \rho_0$.

Since the accuracy of the above approximation depends on the choice of the finite step size $\Delta\rho$, a graph of normalized error (10^{-n} corresponds approximately to n-digit accuracy) incurred in eigenvector derivatives computed by finite differencing was plotted with respect to step size as shown in Fig. 1. It can be seen from the figure that the "optimum" step size for computing the finite-difference derivative is about $\Delta\rho = 0.3 \times 10^{-5}$ and the finite-difference derivative and the derivative computed by the formula "with" diagonal terms correlate to within six decimal places. The finite-difference derivatives can be calculated to five or better digits for $\Delta\rho$ in the interval $10^{-7} < \Delta\rho < 10^{-4}$. From the other experiments, depending upon the local behavior of the eigenvectors and machine word length, we can usually use this experimental approach to find a $\Delta\rho$ range giving five-to-six digit confirmation of Eqs. (16–22). Of course, the importance of the analytical partial derivatives lies in the fact that we do not require the finite-difference approximation and its associated numerical pitfalls and experimentation.

Table 1 shows the right and left eigenvector derivatives computed in three different ways, namely, "without" diagonal terms, "with" diagonal terms, and by finite difference with step size 0.3×10^{-5}. It can be concluded that the eigenvector derivatives computed using the formula "with" diagonal terms shows very accurate agreement (identical, to within small errors in the sixth digit) with the optimized finite-difference approximations. The errors between the finite-difference results and the "without" results are essentially the missing diagonal terms; these frequently occur in the second and third digits. Furthermore, neglecting the diagonal terms in the eigenvector derivative calculation often leads to serious errors, as is evident in Table 1. In summary, the first-order change in

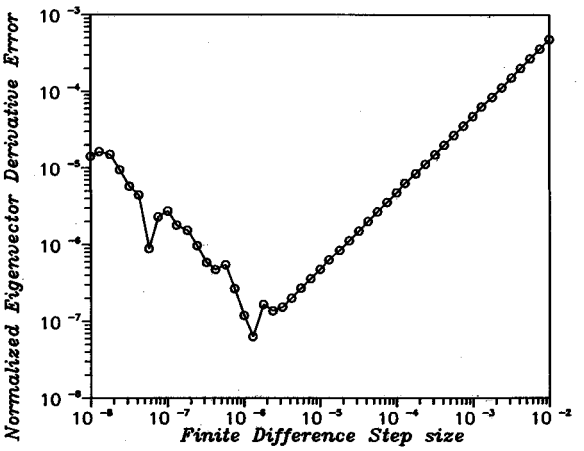

Fig. 1 Normalized error of finite-difference right eigenvector derivative for mode 1.

Table 1 Finite-difference eigenvector derivatives (step size = 0.3E-05)

With diagonal terms

Right

Re(x_1)	Im(x_1)	Re(x_2)	Im(x_2)	Re(x_3)	Im(x_3)	Re(x_4)	Im(x_4)	Re(x_5)	Im(x_5)
−0.0201	0.0000	0.0106	0.0000	−0.0221	0.0271	−0.0221	−0.0271	−0.0247	0.0000
0.0871	0.0000	−0.0206	0.0000	−0.0020	−0.0160	−0.0020	0.0160	0.0159	0.0000
0.0084	0.0000	0.0094	0.0000	−0.0002	0.0015	−0.0002	−0.0015	−0.0159	0.0000
−0.0184	0.0000	0.0042	0.0000	−0.0010	−0.0120	−0.0010	0.0120	−0.0075	0.0000
0.0255	0.0000	0.0039	0.0000	0.0102	0.0156	0.0102	−0.0156	−0.0051	0.0000

Left

Re(y_1)	Im(y_1)	Re(y_2)	Im(y_2)	Re(y_3)	Im(y_3)	Re(y_4)	Im(y_4)	Re(y_5)	Im(y_5)
0.0072	0.0000	0.0020	0.0000	0.0089	0.2608	0.0089	−0.2608	0.0192	0.0000
0.0055	0.0000	0.0069	0.0000	−0.0028	−0.1874	−0.0028	0.1874	0.0042	0.0000
−0.0172	0.0000	−0.0639	0.0000	−0.0210	0.0142	−0.0210	−0.0142	0.0454	0.0000
0.0258	0.0000	−0.0539	0.0000	0.0534	0.0423	0.0534	−0.0423	0.0807	0.0000
0.0081	0.0000	−0.0238	0.0000	0.0153	0.1654	0.0153	−0.1654	0.0496	0.0000

Without diagonal terms

Right

Re(x_1)	Im(x_1)	Re(x_2)	Im(x_2)	Re(x_3)	Im(x_3)	Re(x_4)	Im(x_4)	Re(x_5)	Im(x_5)
−0.0143	0.0000	0.0015	0.0000	−0.0109	0.0144	−0.0109	−0.0144	−0.0434	0.0000
0.0907	0.0000	−0.0262	0.0000	−0.0527	−0.0221	−0.0527	0.0221	0.0010	0.0000
−0.0147	0.0000	0.0063	0.0000	0.0130	0.0043	0.0130	−0.0043	−0.0095	0.0000
−0.0067	0.0000	0.0006	0.0000	0.0191	0.0004	0.0191	−0.0004	−0.0056	0.0000
0.0341	0.0000	0.0102	0.0000	−0.0490	−0.0089	−0.0490	0.0089	0.0161	0.0000

Left

Re(y_1)	Im(y_1)	Re(y_2)	Im(y_2)	Re(y_3)	Im(y_3)	Re(y_4)	Im(y_4)	Re(y_5)	Im(y_5)
0.0020	0.0000	0.0095	0.0000	0.0449	0.1439	0.0449	−0.1439	0.0121	0.0000
−0.0021	0.0000	0.0132	0.0000	0.0037	−0.0944	0.0037	0.0944	0.0129	0.0000
0.0031	0.0000	−0.0500	0.0000	−0.0162	0.0180	−0.0162	−0.0180	0.0138	0.0000
0.0046	0.0000	−0.0358	0.0000	0.0148	0.1128	0.0148	−0.1128	0.0363	0.0000
0.0016	0.0000	−0.0185	0.0000	0.0535	0.1205	0.0535	−0.1205	0.0205	0.0000

an eigenvector has a nonzero projection onto the eigenvector, for non-self-adjoint systems.

The present paper is most concerned with the correction of an assumption made in several completed derivations of equations for the eigenvector derivatives (e.g., as found in Refs. 9, 10, and 20) and resolving an indeterminancy remaining in Ref. 13. A careful re-examination of the role played by the normalizations and biorthogonality conditions for a general non-self-adjoint eigenvalue problem leads to a unique generalization and correction of equations for eigenvector derivatives derived in the previous papers and books cited.

Additionally, Nelson's algebraic method[15] is extended for computing both the right and left eigenvector derivatives. Finally, the improved modal method[17] is extended to the general (non-self-adjoint) eigenvalue problem.

The Eigenvalue Problem

We begin by reviewing familiar results for the non-self-adjoint eigenvalue problem. Let us write the right and left eigenvalue problems as

$$\text{Right:} \quad Ax_j = \lambda_j x_j, \qquad j = 1,\ldots,n \tag{1}$$

$$\text{Left:} \quad y_i^T A = \lambda_i y_i^T, \qquad i = 1,\ldots,n \tag{2}$$

where A is a nondefective $n \times n$ real matrix, x_j and y_j the jth $n \times 1$ complex-valued eigenvectors, and λ_j a generally complex eigenvalue of A.

Premultiplying Eq. (1) by y_i^T and postmultiplying Eq. (2) by x_j, we get

$$y_i^T A x_j = \lambda_j y_i^T x_j \tag{3}$$

$$y_i^T A x_j = \lambda_i y_i^T x_j \tag{4}$$

By subtracting Eq. (4) from Eq. (3), we get

$$0 = (\lambda_j - \lambda_i) y_i^T x_j \tag{5}$$

Restricting ourselves to the class of problems where the eigenvalues are distinct, Eq. (5) leads to the biorthogonality property

$$y_i^T x_j = 0, \qquad i \neq j \tag{6}$$

For $i = j$, the left and right eigenvectors can be normalized such that

$$y_i^T x_i = s_i, \qquad i = 1,\ldots,n \tag{7}$$

where s_i represents chosen normalization constants, commonly set to unity. At this point, we note the fact that biorthogonality of Eq. (6) is a property of the eigensystem and is independent of whatever a priori normalizations (of x_i or y_j) have been imposed, which are also arbitrary.

In matrix notation, Eqs. (4), (6), and (7) can be written as the pair of matrix equations

$$Y^T A X = S\Lambda \tag{8}$$

$$Y^T X = S \tag{9}$$

where $S = \text{diag}(s_1,\ldots,s_n)$, $\Lambda = \text{diag}(\lambda_1,\ldots,\lambda_n)$, and X, Y are $n \times n$ right and left modal matrices (containing x_j and y_i, respectively, as columns). Note that from Eq. (9), we could solve for X or Y since

$$Y = (X^{-1})^T S \tag{10}$$

$$X = (Y^{-1})^T S \tag{11}$$

From the above equations, it is apparent that *after* solving the right eigenvalue problem and imposing arbitrary normalizations on the columns x_j of X, we still need to choose S (i.e., the second set of n normalization constants) to solve for Y from Eq. (10) uniquely. [Incidentally, we note that Eq. (11) of Ref. 14, supposedly providing an expression for an incomplete set of left eigenvectors from a corresponding set of right eigenvectors, requires the inversion of a singular matrix! It is clear that the solution in Ref. 14, for the incomplete set case, is nonunique.] Similarly, if the left eigenvalue problem were solved instead, then X is computed from Eq. (11). In other words, the two sets of eigenvectors, X and Y, require two independent, arbitrary sets of normalizations for their unique representations. We stress here the fact that simply requiring that $S = I$, as is common practice, does not uniquely scale X and Y for non-self-adjoint systems. Henceforth, we let the two sets of normalizations be represented by

$$x_i^T x_i = f_i, \qquad i = 1,\ldots,n \tag{12}$$

$$y_i^T x_i = s_i, \qquad i = 1,\ldots,n \tag{13}$$

where f_i and s_i, $i = 1,\ldots,n$, are fixed normalization constants. We note above that Eq. (12) amounts to a generalization of the normalization approach of Ref. 14 where the element corresponding to a displacement coordinate with the largest modulus is set to unity.

Eigenvalue Derivatives

For completeness, we rederive the eigenvalue derivatives with respect to a scalar parameter ρ. First, we take partial derivatives of Eq. (1) to get

$$\frac{\partial A}{\partial \rho} x_j + A \frac{\partial x_j}{\partial \rho} = \frac{\partial \lambda_j}{\partial \rho} x_j + \lambda_j \frac{\partial x_j}{\partial \rho} \tag{14}$$

We then premultiply the above equation by y_j^T to obtain

$$y_j^T \frac{\partial A}{\partial \rho} x_j + y_j^T (A - \lambda_j I) \frac{\partial x_j}{\partial \rho} = \frac{\partial \lambda_j}{\partial \rho} y_j^T x_j$$

so that by Eq. (2), the eigenvalue derivative expression takes the form

$$\frac{\partial \lambda_j}{\partial \rho} = \frac{1}{s_j} \left(y_j^T \frac{\partial A}{\partial \rho} x_j \right) \tag{15}$$

where s_j is the jth normalization constant chosen in Eq. (13).

Eigenvector Derivatives

Here, we derive an expression for eigenvector derivatives along the lines of Ref. 13 by using $\{x_1,\ldots,x_n\}$ as basis vectors,

$$\frac{\partial x_i}{\partial \rho} = \sum_{j=1}^{n} \alpha_{ij} x_j \tag{16}$$

Since the eigenvalues were assumed distinct, this always leads to a set of n eigenvectors that are linearly independent and that may be used as a set of basis vectors for spanning the complex n-dimensional space. To obtain expansion coefficients α_{ij}, $i, j = 1,\ldots,n$, we begin by substituting Eq. (16) into Eq. (14) and premultiply by y_k^T to get

$$y_k^T \frac{\partial A}{\partial \rho} x_i + \sum_{j=1}^{n} \alpha_{ij} \lambda_j y_k^T x_j = \frac{\partial \lambda_i}{\partial \rho} y_k^T x_i + \lambda_i \sum_{j=1}^{n} \alpha_{ij} y_k^T x_j$$

and by using biorthogonality conditions, we obtain

$$(\lambda_k - \lambda_i)\alpha_{ik} y_k^T x_k = \frac{\partial \lambda_i}{\partial \rho} y_k^T x_i - y_k^T \frac{\partial A}{\partial \rho} x_i$$

Since A was assumed to have distinct eigenvalues, we can solve for α_{ik} as

$$\alpha_{ik} = -\frac{1}{(\lambda_k - \lambda_i)s_k}\left(y_k^T \frac{\partial A}{\partial \rho} x_i\right), \qquad k \neq i \qquad (17)$$

We note that, for computational efficiency, the matrix-vector multiplication required in Eq. (15) could be stored for use in Eq. (17). For the case of $i = k$, α_{ii} can be computed from the normalization condition of Eq. (12). By taking its partial derivatives and using Eq. (16), we find

$$\frac{\partial x_i^T}{\partial \rho} x_i = 0$$

$$\alpha_{ii} x_i^T x_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_{ij} x_j^T x_i = 0$$

which leads to the expression

$$\alpha_{ii} = -\frac{1}{f_i} \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_{ij} x_j^T x_i \qquad (18)$$

where f_i is the ith normalization constant chosen in Eq. (12). It can be deduced from Eq. (18) that the diagonal terms α_{ii} become identically zero if: 1) matrix A has orthogonal eigenvectors (for example, A is real and symmetric) or 2) the ith eigenvector is insensitive with respect to a particular parameter except in the ith eigenvector direction. We also observe from Eqs. (17) and (18) that the off-diagonal and diagonal coefficients α_{ik} and α_{ii} depend on the normalization constants s_k and f_i, respectively.

To obtain left eigenvector derivatives, we similarly let

$$\frac{\partial y_i}{\partial \rho} = \sum_{j=1}^{n} \gamma_{ij} y_j \qquad (19)$$

By taking partial derivatives of Eq. (2), substituting Eq. (19), and postmultiplying by x_k, we obtain

$$\sum_{j=1}^{n} \gamma_{ij} y_j^T A x_k + y_i^T \frac{\partial A}{\partial \rho} x_k = \frac{\partial \lambda_i}{\partial \rho} y_i^T x_k + \lambda_i \sum_{j=1}^{n} \gamma_{ij} y_j^T x_k$$

By using Eqs. (1) and (6), we get

$$\gamma_{ik}\lambda_k y_k^T x_k + y_i^T \frac{\partial A}{\partial \rho} x_k = \frac{\partial \lambda_i}{\partial \rho} y_i^T x_k + \lambda_i \gamma_{ik} y_k^T x_k$$

$$(\lambda_k - \lambda_i)\gamma_{ik} y_k^T x_k = \frac{\partial \lambda_i}{\partial \rho} y_i^T x_k - y_i^T \frac{\partial A}{\partial \rho} x_k$$

$$\gamma_{ik} = -\frac{1}{(\lambda_k - \lambda_i)s_k}\left(y_i^T \frac{\partial A}{\partial \rho} x_k\right), \qquad k \neq i \qquad (20)$$

By comparing Eqs. (17) and (20), we observe the antisymmetric property

$$\gamma_{ik} = -\alpha_{ki}, \qquad k \neq i \qquad (21)$$

For $k = i$, γ_{ii} and α_{ii} are related through the normalization relation of Eq. (13). Thus, by taking the partial derivative of Eq. (13), we get

$$\frac{\partial y_i^T}{\partial \rho} x_i + y_i^T \frac{\partial x_i}{\partial \rho} = 0$$

and by using the expansions of Eqs. (16) and (19)

$$\sum_{j=1}^{n} \gamma_{ij} y_j^T x_i + y_i^T \sum_{j=1}^{n} \alpha_{ij} x_i = 0$$

$$\gamma_{ii} y_i^T x_i + \alpha_{ii} y_i^T x_i = 0$$

$$\gamma_{ii} = -\alpha_{ii} \qquad (22)$$

From Eqs. (21) and (22), we observe the nice relationship between the left and right eigenvector derivative expansion coefficients,

$$[\gamma] = -[\alpha]^T \qquad (23)$$

provided, of course, that the normalization constants s_k are chosen to have the same values. The significance of Eq. (23) is clear: the left and right eigenvector derivatives map into each other uniquely.

The above procedure can be extended to the generalized eigenvalue problem of the form

$$Ax_i = \lambda_i B x_i$$

$$A^T y_i = \lambda_i B^T y_i \qquad i = 1, \ldots, n$$

with biorthogonality and normalizations

$$y_k^T B x_i = \delta_{ik}, \qquad i, k = 1, \ldots, n$$

$$x_i^T B x_i = 1, \qquad i = 1, \ldots, n$$

After some algebra analogous to the above developments, it can be shown that the eigenvalue and eigenvector derivatives take the form

$$\frac{\partial \lambda_i}{\partial \rho} = y_i^T\left(\frac{\partial A}{\partial \rho} - \lambda_i \frac{\partial B}{\partial \rho}\right) x_i$$

$$\frac{\partial x_i}{\partial \rho} = \sum_{k=1}^{n} \alpha_{ik} x_k, \qquad \frac{\partial y_i}{\partial \rho} = \sum_{k=1}^{n} \gamma_{ik} y_k$$

where

$$\alpha_{ik} = \frac{y_k^T\left(\frac{\partial A}{\partial \rho} - \frac{\partial \lambda_i}{\partial \rho} B - \lambda_i \frac{\partial B}{\partial \rho}\right) x_i}{(\lambda_i - \lambda_k)}, \qquad i \neq k$$

$$= -\frac{1}{2}\left[x_i \frac{\partial B}{\partial \rho} x_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_{ij} x_j^T (B + B^T) x_i\right], \qquad i = k \qquad (24)$$

$$\gamma_{ik} = \frac{y_i^T\left(\frac{\partial A}{\partial \rho} - \frac{\partial \lambda_i}{\partial \rho} B - \lambda_i \frac{\partial B}{\partial \rho}\right) x_k}{(\lambda_i - \lambda_k)}, \qquad i \neq k$$

$$= -y_i^T \frac{\partial B}{\partial \rho} x_i - \alpha_{ii}, \qquad i = k \qquad (25)$$

It is interesting to note that for the special case where A and B are real and symmetric, the eigenvector derivative coefficients reduce, as a consequence of orthogonality, to the forms originally derived in Ref. 12. Furthermore, if B is in addition constant (with respect to ρ's variation)—for exam-

ple, the standard symmetric eigenvalue problem where $B = I$ —then the diagonal coefficients vanish and this conforms to the standard symmetric case found in Refs. 9 and 10. However, if A is nonsymmetric, the eigenvectors are not generally orthogonal and the diagonal coefficients from Eqs. (24) and (25) are clearly nonzero, even when B is a constant. This crucial result directly contradicts the reasoning in Ref. 20 where the diagonal coefficients are noted to be "arbitrary" and hence the "most convenient" choice is zero, which is clearly in error! This result is also interesting in the context of a first-order eigenvector perturbation analysis since it implies that, in the *modal* expansion of a particular eigenvector, the contribution of the change in the eigenvector in the same direction cannot be assumed zero in general for non-self-adjoint problems, as done commonly, for example in Ref. 10, but subsequently corrected in Ref. 22.

An Improved Modal Method

Equation (16) for eigenvector derivatives requires the knowledge of all n eigenvectors. For large systems, only the lowest L pairs of eigensolutions may be computed accurately, where $L \ll n$. In this case, an improved modal method, analogous to the modal truncation method reported in Ref. 17, can be derived, provided matrix $[A]$ is nonsingular.

Rewrite Eq. (16) as

$$\frac{\partial x_i}{\partial \rho} = \alpha_{ii} x_i + z_i \tag{26}$$

where

$$z_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_{ij} x_j \tag{27}$$

Using Eq. (24), Eq. (27) can be written as

$$z_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{y_j^T F_i}{\lambda_i - \lambda_j} x_j$$

or

$$z_i = \sum_{\substack{j=1 \\ j \neq i}}^{L} \frac{y_j^T F_i}{\lambda_i - \lambda_j} x_j + \sum_{j=L+1}^{n} \frac{y_j^T F_i}{\lambda_i - \lambda_j} x_j \tag{28}$$

where

$$F_i = \left(\frac{\partial A}{\partial \rho} - \frac{\partial \lambda_i}{\partial \rho} B - \lambda_i \frac{\partial B}{\partial \rho} \right) x_i \tag{29}$$

and we have assumed that $i \leq L$. If the eigenvalues are numbered according to their magnitude in ascending order, then for the class of problems with a large frequency gap is

$$\lambda_j - \lambda_i \approx \lambda_j \quad \text{for } j > L$$

It is clear that the above approximation is very accurate for $j \gg L$. Thus, Eq. (28) can be written, letting z_i be approximated by $\bar{z}_i$, as

$$\bar{z}_i = \sum_{\substack{j=1 \\ j \neq i}}^{L} \frac{y_j^T F_i}{\lambda_i - \lambda_j} x_j + \sum_{j=L+1}^{n} \frac{y_j^T F_i}{-\lambda_j} x_j \tag{30}$$

which can be rewritten as

$$\bar{z}_i = \sum_{\substack{j=1 \\ j \neq i}}^{L} \frac{y_j^T F_i}{\lambda_i - \lambda_j} x_j + \sum_{j=1}^{n} \frac{y_j^T F_i}{-\lambda_j} x_j - \sum_{j=1}^{L} \frac{y_j^T F_i}{-\lambda_j} x_j \tag{31}$$

The biorthogonality conditions can be written in matrix form as

$$[Y]^T [A][X] = [\Lambda] \tag{32}$$

or

$$[X]^{-1}[A]^{-1}([Y]^T)^{-1} = [\Lambda]^{-1}$$

Thus,

$$[A]^{-1} = [X][\Lambda]^{-1}[Y]^T \tag{33}$$

or

$$[A]^{-1} = \sum_{j=1}^{n} \frac{x_j y_j^T}{\lambda_j} \tag{34}$$

Using the spectral decomposition of Eq. (34), the second summation on the right-hand side of Eq. (31) becomes $-A^{-1}F_i$ and Eq. (31) can be written as

$$\bar{z}_i = \sum_{\substack{j=1 \\ j \neq i}}^{L} \frac{y_j^T F_i}{\lambda_i - \lambda_j} x_j - A^{-1} F_i + \sum_{j=1}^{L} \frac{y_j^T F_i}{\lambda_j} x_j \tag{35}$$

We note that the $A^{-1}F_i$ term in Eq. (35) approximates the effect of higher-order modal components on eigenvector derivatives of lower-order modes. Presumably, for many cases, this approximation is more accurate than an initial expansion involving only the first L modes. Now the modal representation of eigenvector derivative can be approximated as

$$\frac{\partial x_i}{\partial \rho} = \bar{\alpha}_{ii} x_i + \bar{z}_i \tag{36}$$

where $\bar{\alpha}_{ii}$ can be computed by requiring

$$\frac{\partial}{\partial \rho}\left(x_i^T B x_i \right) = 0$$

which leads to

$$\bar{\alpha}_{ii} = -\frac{1}{2}\left(x_i^T \frac{\partial B}{\partial \rho} x_i + x_i^T B \bar{z}_i + \bar{z}_i^T B x_i \right) \tag{37}$$

For the class of problems that satisfy the previously mentioned frequency conditions, the modal approximation of Eq. (36) may represent a significant improvement over Eq. (16), vis-a-vis computation, and often provides acceptable precision. For problems without well-separated eigenvalues, similar approximations can be developed with the summation of Eq. (16) truncated after a few terms.

For self-adjoint eigenvalue problems in structural dynamics, this improved modal approximation approach has been demonstrated to be efficient and accurate.[18] Similarly, the derivatives of the left eigenvectors can be computed using the following improved modal approximation:

$$\frac{\partial y_i}{\partial \rho} = \bar{\gamma}_{ii} y_i + \bar{w}_i \tag{38}$$

where

$$\bar{w}_i = \sum_{\substack{j=1 \\ j \neq i}}^{L} \gamma_{ij} y_i - A^{-1} G_i + \sum_{j=L+1}^{n} \frac{G_i^T x_j}{\lambda_j} y_j \tag{39}$$

$$G_i^T = y_i^T\left(\frac{\partial A}{\partial \rho} - \frac{\partial \lambda_i}{\partial \rho}B - \lambda_i\frac{\partial B}{\partial \rho}\right) \tag{40}$$

$$\bar{\gamma}_{ii} = -y_i^T\frac{\partial B}{\partial \rho}x_i - \bar{\alpha}_{ii} - \bar{w}_i^T B x_i - y_i^T B\bar{z}_i \tag{41}$$

and γ_{ij} is given by Eq. (25).

Algebraic Method for Computing Eigenvector Derivatives

Nelson[15] has proposed an algebraic method for computing eigenvector derivatives. In this formulation, the only information needed is the eigenvector being differentiated. Following the approach used in Ref. 15, we summarize equations for computing $\partial x_i/\partial \rho$ and $\partial y_i/\partial \rho$.

Right eigenvector:

$$\frac{\partial x_i}{\partial \rho} = \alpha_i x_i + V_x \tag{42}$$

where V_x is the solution of

$$\bar{Z}V_x = -\bar{F}_i \tag{43}$$

and where $\bar{Z}$ is matrix Z with the rth column and rth row replaced by e_r and e_r^T, respectively; e_r is a vector containing one in element r and zeroes elsewhere, $Z = A - \lambda_i B$, $\bar{F}_i$ is vector F_i with the rth row replaced by 0, F_i is given by Eq. (29), and

$$\alpha_i = -\frac{1}{2}\left(x_i^T\frac{\partial B}{\partial \rho}x_i + V_x^T B x_i + x_i^T B V_x\right) \tag{44}$$

Left eigenvector:

$$\frac{\partial y_i}{\partial \rho} = \gamma_i y_i + V_y \tag{45}$$

where V_y is the solution of

$$\bar{Z}^T V_y = -\bar{G}_i \tag{46}$$

$\bar{Z}^T$ = matrix Z^T, with rth column and rth row replaced by e_r and e_r^T, respectively; $G_i = (\partial/\partial\rho)(Z^T)y_i$, $\bar{G}_i$ is vector G_i with the rth row replaced by a row of zeros, and

$$\gamma_i = -y_i^T\frac{\partial B}{\partial \rho}x_i - \alpha_i - V_y^T B x_i - y_i^T B V_x \tag{47}$$

In the above formulation, the value r is chosen to correspond to the maximum component of x_i (or y_i) in order to achieve computational robustness.

Using the above formulations, the eigenvector derivatives can be computed using only the eigenvector of interest together with some algebraic manipulations. The success of this method hinges on the existence of a well-conditioned nonsingular $\bar{Z}$. A more general algebraic method, which eliminates this restriction, has been reported by Chen and Wei.[16] Their formulation is based upon the matrix decomposition and generalized inverse techniques.

The present approach can be extended to the case of repeated eigenvalues. This has already been done for self-adjoint systems by Ojalvo.[21] In the next section, we present a numerical example to demonstrate the errors incurred in neglecting the diagonal terms in the evaluation of eigenvector derivatives via modal expansion for a non-self-adjoint problem.

Conclusions

We have highlighted the need for two independent sets of normalizations to uniquely define the right and left eigenvector sets. This is crucial in the complete formulation of eigenvector derivatives via modal expansion. In addition, a concise equation relating left and right eigenvector derivative expansion coefficients is given. We have also demonstrated beyond any doubt that the first-order change in an eigenvector has a generally nonzero projection onto the eigenvector, for non-self-adjoint systems.

For many non-self-adjoint systems (encountered commonly, for example, in linear control theory), the matrices are fully populated. The consequence is that algebraic methods that take advantage of the bandedness patterns of the matrices become less attractive than modal expansion methods, where solutions of auxiliary linear equations are not needed other than solution of the eigenvalue problem itself. Additionally, since no auxiliary linear equations need to be solved, the associated possibility of rank deficiency is avoided. On the other hand, the theoretical problem of requiring "all" eigenvectors in modal expansion methods may turn out in many applications not to pose a practical difficulty.

These latter issues cannot be resolved completely on the basis of analytical arguments. Their resolution requires numerical analysis in the context of particular applications.

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